On this chord are found two projective quadruples of points:

$$\begin{array}{llll} P,\ Q, & \text{one in } AA_1C, & \text{one in } AA_1C_1; \\ Q,\ P, & \text{“ “ } BB_1C_1, & \text{“ “ } BB_1C. \end{array}$$

Owing to the double correspondence of P and Q, these are three pairs in involution. Aside from P and Q, two points of a pair are found in two planes determined by mutually exclusive triads taken from the given 6 points.

As any two chords can be chosen for axes, we may use AA_1 and B_1C_1. The resulting tetrads of points on PQ are then:

$$\begin{array}{llll} P,\ Q, & \text{one in } AA_1C, & \text{one in } AA_1B; \\ Q,\ P, & \text{“ “ } B_1C_1B, & \text{“ “ } B_1C_1C. \end{array}$$

From the first three pairs it is found that this involution is identical with the former, hence the fourth pair lies in the same involution. That fourth pair, in planes AA_1B and CC_1B_1, is determined like the former pairs by two mutually exclusive (or supplementary) triads of points in the given sextette. By repetition of this permutation process we can show that any desired pair of supplementary triads give, on the chord PQ, a pair in this same involution. In all, there are ten such pairs of planes, giving ten pairs of points in involution in addition to the pair P, Q.

As Serret points out, it is sufficient to show that the eight faces of a simple octahedron on the six points cut a line in four pairs of points in involution, and from this can be inferred the remainder. Hence there may be written down two equations in line-coördinates which are satisfied by chords (double secants) of a gauche cubic through the six vertices of the octahedron.

A CERTAIN TWO-DIMENSIONAL LOCUS.

By J. L. WALSH, Harvard University.

The writer has recently had occasion to consider the following problem, in connection with the approximate determination of the roots of certain types of polynomials:[1] If two points z_1 and z_2 have as their respective loci the interiors (boundaries included) of two circles, determine the locus of a point z given by the relation $z = (m_2z_1 + m_1z_2)/(m_1 + m_2)$, when m_1 and m_2 are real or complex constants. A closely allied problem is found by considering the locus of the point z determined as before, but where in addition m_1 and m_2 also vary, and take all

[1] See several papers already published and others about to be published in the *Transactions of the American Mathematical Society.*

The following interpretation can be given to the problem of the present note: Let the points $\alpha_1, \alpha_2, \cdots \alpha_n$ vary independently and have as common locus the interior and boundary of a circle C_1, and let the points $\beta_1, \beta_2, \cdots \beta_n$ vary independently and have as common locus the interior and boundary of a circle C_2; determine the locus of the roots of the equation

$$(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) = A(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)$$

where A takes all the values such that $|A| = 1$.

values such that the relation $|m_1| = |m_2|$ is satisfied.[1] Otherwise expressed, the problem is to determine the locus of a point z which is equidistant from points z_1 and z_2 varying as described. It is the purpose of this note to present a solution of that problem; the answer is contained in the

THEOREM. *Let the loci of two points z_1 and z_2 be, respectively, the interiors (boundaries included) of two circles C_1 and C_2. Then the locus of a point z, equidistant from z_1 and z_2, is either the entire plane or the exterior of a certain hyperbola whose foci are the centers A_1 and A_2 of C_1 and C_2, according as the loci of z_1 and z_2 have or have not a point in common.*

We understand by the *exterior* of a hyperbola the region of the plane bounded by and lying between the two branches of the hyperbola, including the points of the hyperbola itself. The *locus* of z is taken to consist of all points equidistant from two points z_1 and z_2 satisfying the given conditions.

If C_1 and C_2 are not entirely external to each other—that is, if the loci of z_1 and z_2 have a point in common—the two points z_1 and z_2 may be chosen to coincide. Then any point of the plane is equidistant from z_1 and z_2 chosen at this common position, so every point of the plane is a point of the locus.

If the loci of z_1 and z_2 have no common point, C_1 and C_2 are mutually external. There are some points of the plane, such as on the perpendicular bisector of the segment A_1A_2, which belong to the locus of z. There are some points, such as on the line A_1A_2 but not on the segment A_1A_2, which do not belong to that locus; for *all* the points z_1 (or z_2) are nearer to one of these latter points than *any* point z_2 (or z_1). We must determine the boundary separating the points z of the locus from the points not of the locus.

Let z' be a point on the boundary of the locus of z and be equidistant from points z_1' and z_2' in their proper loci. Then z_1' must be actually on C_1, for otherwise we could move z_1 all over a small area interior to C_1 surrounding z_1', and we could consider z to be determined from z_1 and z_2' so that the triangle z_1zz_2' remains constantly similar to the triangle $z_1'z'z_2'$. Then z would move over a small area completely surrounding z', every point of this small area would be a point of the locus of z, and hence z' could not be on the boundary of that locus. We know, then, that z_1' and z_2' must lie on C_1 and C_2 respectively.

Moreover, z_1' and z_2' must lie on C_1 and C_2 in such a way that z_1' is the point of C_1 nearest to z' and z_2' the point of C_2 farthest from z', or so that z_1' is the point of C_1 farthest from z' and z_2' the point of C_2 nearest to z'. Thus, if z_1' satisfies neither of these conditions, there can be chosen a point z_1'' *interior to* C_1 but such that the distances $z'z_1''$ and $z'z_1'$ (and hence $z'z_1''$ and $z'z_2'$) are equal, so by the reasoning already given z' is not on the boundary of the locus of z. If z_1' and z_2' are the points of C_1 and C_2 farthest from z', there are two points z_1'' and z_2'' interior to C_1 and C_2, respectively, and on the lines $z'z_1'$ and $z'z_2'$ which are

[1] If we consider the allied problems using the relations $|m_1/m_2| = \rho$, a constant, or $(m_1/m_2)/|m_1/m_2| = e^{i\phi}$, a constant, we are led in the first case to a locus which is a doubly connected region bounded by a quartic, and in the second case to a simply connected region bounded by a curve of the eighth degree. The precise equations of these boundaries may easily be found by the methods of this note.

equidistant from z', so z' is not on the boundary of the locus of z. Similarly we may show that z_1' and z_2' cannot be the points of C_1 and C_2 nearest to z', so z_1' and z_2' must satisfy the conditions stated.

The points z', z_1', A_1 are collinear and similarly the points z', z_2', A_2. The point A_1 is on the segment $z'z_1'$ if and only if A_2 is not on the segment $z'z_2'$. The distances $z'z_1'$ and $z'z_2'$ are equal by hypothesis, so the distances $z'A_1$ and $z'A_2$ differ by the sum of the distances A_1z_1' and A_2z_2', that is, by the sum of the radii of C_1 and C_2. Then z' lies on the hyperbola whose foci are A_1 and A_2 and whose "constant difference" is the sum of the radii of C_1 and C_2. The locus of z is not the entire plane and therefore has a boundary; the locus contains the perpendicular bisector of A_1A_2 but no point of the line A_1A_2 not on the finite segment A_1A_2. Hence this locus must be bounded by the entire hyperbola and is the exterior of the hyperbola. This completes the proof of the theorem.

Denote by B_1' and B_1'' and by B_2' and B_2'' the intersections of the line A_1A_2 with C_1 and C_2, respectively, determined so that B_1' separates B_1'' and B_2'' but B_2' does not separate B_1'' and B_2''. The hyperbola cuts the line A_1A_2 at the midpoints of the segments $B_1'B_2'$ and $B_1''B_2''$. The reader will easily prove that the asymptotes of the hyperbola are the perpendicular bisectors of the transverse tangents to C_1 and C_2.

For the problem just considered, the point z' can never lie on the segment $z_1'A_1$; otherwise z_2' would be within C_1, and C_1 and C_2 are supposed to be mutually external. But if we modify our problem by assigning to z_1 as locus the region of the plane *exterior to* C_1, and if the locus of z is not the entire plane, the point z' always lies between A_1 and z_1'. The locus of z can be shown to be the region of the plane exterior to a certain ellipse whose foci are A_1 and A_2.

If we modify our problem by assigning to z_1 as locus a *half plane*, while the locus of z_2 remains the interior of C_2, the locus of z is either the entire plane or the exterior of a parabola whose focus is A_2 and whose directrix is parallel to the boundary of the locus of z_1.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

20. Babbage Visits Mme. Laplace.

Sir John Herschel, speaking of the status of mathematics and astronomy in Great Britain at the opening of the nineteenth century, remarked that "Mathematics were at the last gasp, and Astronomy nearly so." It was for this reason that he, in conjunction with George Peacock and Charles Babbage, formed the so-called "Analytical Society", the purpose of which was to introduce into Cambridge the Continental type of the calculus and, in general, to revivify the mathematics of England. The same three scholars were influential in establishing the Astronomical Society of London, and each was among the leaders in other efforts of a similar nature, one of Babbage's most important papers being entitled "Reflections on the Decline of Science in England" (1830).